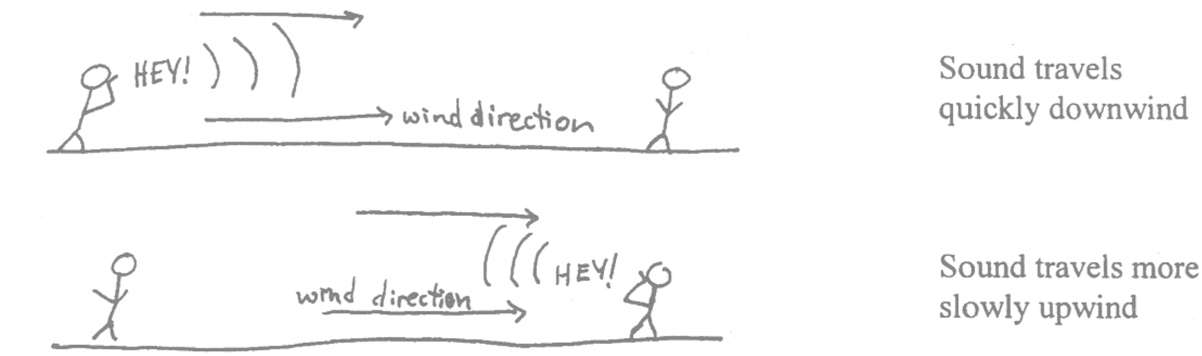
Special Relativity

Walter F. Smith, Haverford College 4-9-04

Propagation of waves through a medium

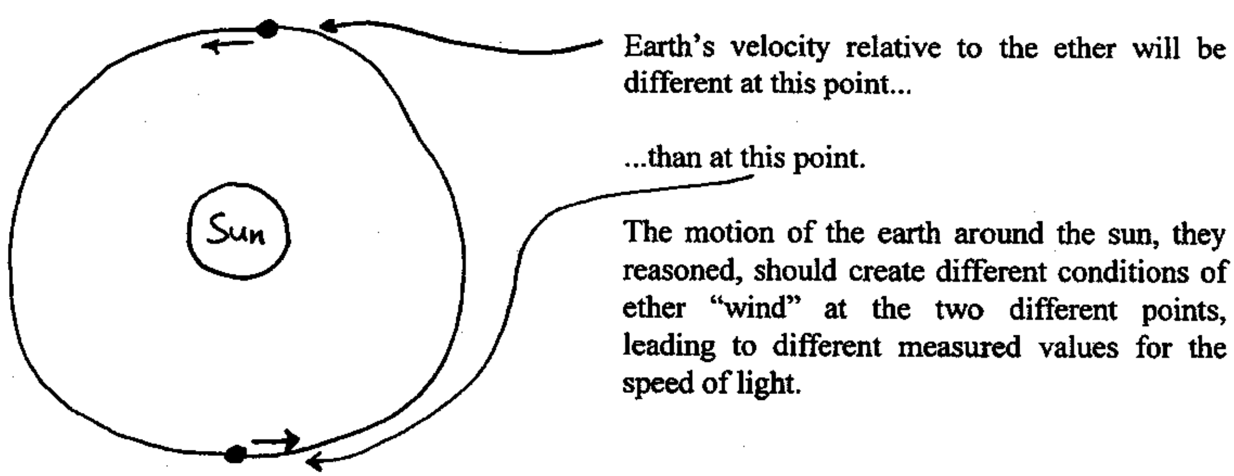
As you'll recall from last semester, when the speed of sound is measured relative to the ground, it is faster for sound waves traveling downwind than upwind:



The air is the "medium" which carries the sound.

The Michaelson-Morley experiment

In the late 1800's, it was universally believed that light waves traveled through a medium as well; the medium was called the "ether." It was assumed that the ether was at rest with respect to the center of the universe, or perhaps the center of the galaxy. In 1887, A. A. Michaelson and E. W. Morley set out to measure differences in the speed of light caused by the motion of the earth relative to the ether:



They made extremely precise measurements, and always found exactly the same value for the speed of light. The only logical conclusion was that light does not need a medium to travel through! It can travel through vacuum! Another way of saying this is that the speed of light measured by any experimenter will always be the same, whether the experimenter is moving to the right, to the left, or is still. In fact, as we've seen, Maxwell's equations show that the propagation of light is a basic form of electromagnetism, which propagates at a speed

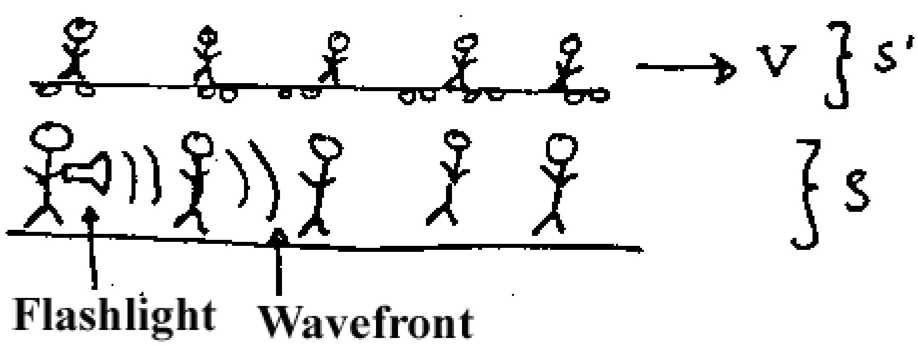
$c = 1/\sqrt{\varepsilon_o \mu_o}$. Note that the speed of the observer doesn't appear in this equation, just as it doesn't appear in $F = ma$. So, in the same way that $F = ma$ works in all constant velocity (or "inertial") reference frames, $c = 1/\sqrt{\varepsilon_o \mu_o}$ works in all reference frames, i.e., light propagates with the same speed in all reference frames. (Again, this was shown by the Michaelson-Morley experiment.)

The basic postulate of relativity:

> *The laws of physics work equally well in all inertial reference frames. There is no preferred reference frame.*

This includes the propagation of light, since as we discussed above, light propagation is a consequence of Maxwell's equations, and since the Michaelson-Morley experiment showed that light propagates at the same speed in a variety of reference frames.

This postulate has immediate counterintuitive consequences. For example, imagine two groups of observers. One group (S) is "stationary," while the other group (S') moves to the right at speed V = 0.9 times the speed of light, i.e., $V = 0.9\ c$. One of the S observers turns on a flashlight, and the other S observers measure the speed at which the wavefront propagates:



The S observers, of course, measure a speed of c for the wavefront. What speed do the S' observers measure? In the way we're accustomed to think, they would measure a speed of $1.0\ c - 0.9\ c = 0.1\ c$. However, this is wrong. The light is propagating in the S' frame as well as the S frame, so it must move with a speed of c in S' as well as in S!

In fact, as we'll see, this strange way of adding velocities is not unique to light propagation. Similar effects occur for any object moving very close to the speed of light.

For example, we will show that if the observers in S throw a rock to the right with speed (measured in S) of $0.99\ c$, then the observers in S' will measure a speed of $0.83\ c$ for the rock, instead of the speed of $0.09\ c$ that one might expect. (You can see that the effect is most extreme for the case of something propagating at exactly c, since both sets of observers measure the same speed for it, despite their large relative velocities, while on the example we just did they measure similar but not identical velocities ($0.99\ c$ in S and $0.83\ c$ in S').

The Galilean Transformation

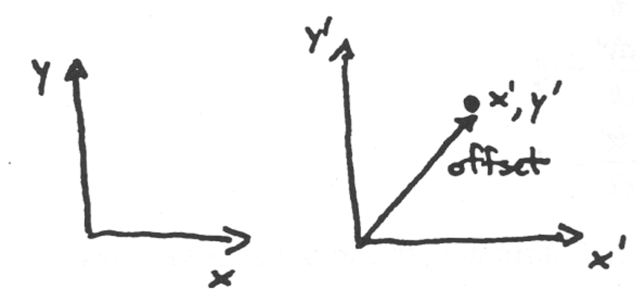
Before we begin with the detailed discussion which leads to the rule used above for velocity addition, as well as to the other very strange relativistic effects you may have heard of (e.g., time dilation, length contraction), let's return to the intuitive arena where things move at speeds much less than c, and examine how we can express coordinates of an event as measured in in S in terms of the coordinates as measures in S'. By "event" I mean simply something with well-defined x, y and z coordinates and a well-defined time when it happens. Good examples include an explosion, a collision, and a particular tick on a particular clock.

Let's say an event occurs at coordinates x', y', z', and t' in the S' reference frame. (Again, S' moves to the right with speed V relative to S.) What are the coordinates x, y, z and t of the event as measured by the observers in x? We assume as we will for the remainder of our treatment of relativity that the origins ($x = y = z = 0$) of the two reference frames coincide at $t = t' = 0$ (as measured by clocks at the origins). We lose no generality by doing this, since we can always choose where $t = 0$. Because of the synchronization, we immediately have that $t = t'$.

Let's start with an easy case: $x' = y' = z' = 0$, i.e., the event occurs at the origin of S' at time t'. Since S' moves to the right with speed V, at time $t = t'$, the S' origin is at

$$x = Vt = Vt', \quad y = 0, \quad z = 0.$$

Now consider an event that occurs someplace else in S', at coordinates x', y', z', t'. This is really just like the case we just considered, except now there is an offset relative to the S' origin of x', y', z':



So, the coordinates in S are offset by the same amount, i.e.,

$$\boxed{\begin{aligned} x &= x' + Vt' \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}}$$

These relations between the event coordinates x', y', z', t' as measured on S' and those as measured on S is called the "Galilean transformation." It's really nothing new, but just a formal way of writing what you already understand about things moving at relatively small speeds.

The Galilean Velocity Transformation

A simple consequence of the Galilean transformation is the velocity addition rule which you're used to, as we'll show here. We consider not just a single event which occurs at x', y', z', t', but rather an object which is moving, i.e., its coordinates x', y', and z' depend on time. The components of its velocity, as measured in the S' frame, are found as usual by taking the derivatives with respect to time: $u'_x \equiv \frac{dx'}{dt'}$ $u'_y \equiv \frac{dy'}{dt'}$ $u'_z \equiv \frac{dz'}{dt'}$. By taking the derivative of the Galilean transformation with respect to t', we can find the relationship between these velocities (measured in S') and those measured in S:

$$\left.\begin{array}{l} x = x' + Vt' \\ y = y' \\ z = z' \end{array}\right\} \xrightarrow{\frac{d}{dt'}} \begin{cases} \dfrac{dx}{dt'} = \dfrac{dx'}{dt'} + V = u'_x + V \\ \dfrac{dy}{dt'} = \dfrac{dy'}{dt'} = u'_y \\ \dfrac{dz}{dt'} = \dfrac{dz'}{dt'} = u'_z \end{cases}$$

Since $t = t'$, we have that $dt = dt'$, so $u_x \equiv \dfrac{dx}{dt} = \dfrac{dx}{dt'}$, etc. Substituting this into the above gives the Galilean velocity transformation:

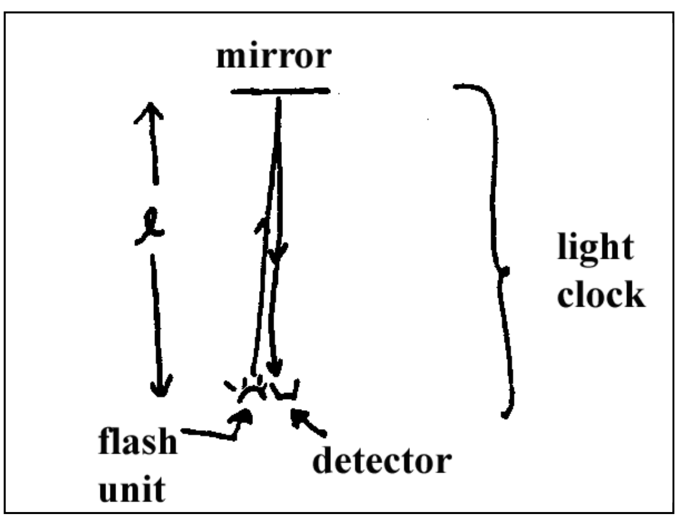
$$u_x = u'_x + V$$
$$u_y = u'_y$$
$$u_z = u'_z$$

This should make intuitive sense to you, but as we've just discussed, it doesn't work when the speeds involved are close to c.

Our Goal
Our goal is to develop the correct versions of the Galilean transformations, versions which work both at ordinary speeds and at speeds close to c. Along the way, we will discover some pretty odd effects:

- The rate at which time passes is not the same in S as it is in S'.
- Events which occur simultaneously in S don't usually occur simultaneously in S'.
- An object has a different length if measured in S than in S'

Time Dilation
The light clock
Part of the reason that velocities close to c don't add in the way we expect is that time is perceived differently in S and S'. To investigate this, we'll use an unusual clock, the "light clock":



A device sends out a flash of light which travels upward, bounces off a small mirror, and then returns to a detector, which is right next to the flash unit. As soon as this detector sees the reflected light flash, it triggers another flash. Each of these cycles is one "tick" of the light clock. Let's put one of these light clocks in the "moving" frame S'. We'll show that the rate at which this clock ticks, as perceived by the observers in S, depends on V, the relative velocity between S' and S.

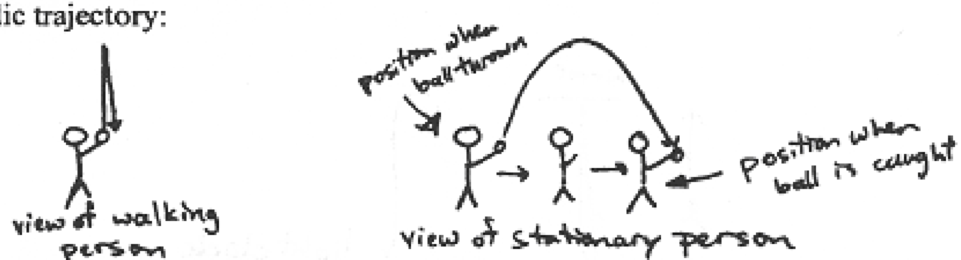
Light clocks vs. ordinary clocks

It’s important to realize that the results we’ll get are not limited to light clocks; any other clock in S’ would display exactly the same variation. To see this, assume that the person in S’ is initially at rest relative to S. She has a conventional clock, which she adjusts so that it has the same tick rate as the light clock. Now she starts moving. Since there is no preferred reference frame (by the basic postulate of relativity), there should be no way for her to tell that she was previously stationary and is now moving, rather than the other way around. For example, since the regular clock and the light clock were synchronized when she was “stationary,” they should remain synchronized now that she is “moving.” We could make a similar argument using her heartbeat. If there are a certain number of ticks of the lightclock per heartbeat when she is “stationary,” there must be the same number when she is moving. It is still possible that the rate at which all these clocks tick (the light clock, the regular clock, and her heart) might vary in *unison*, as seen by the people in S. All that the person in S’ can tell is that the clocks remain synchronized.

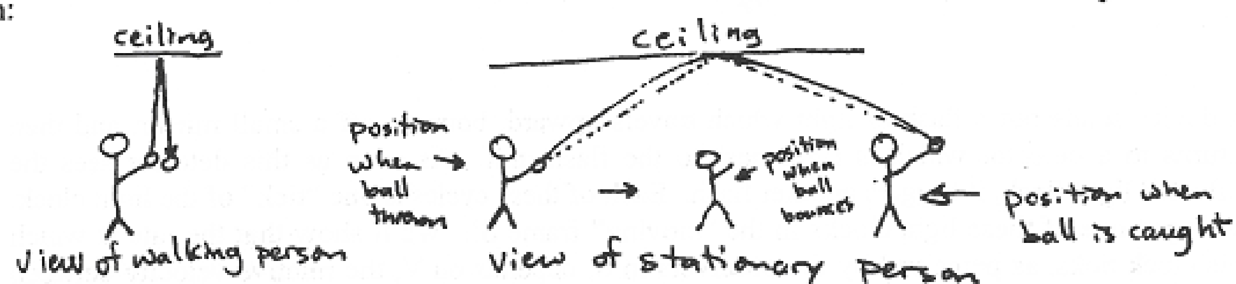
Since we could make these arguments using a chemical reaction or any other time-dependent phenomenon instead of her heartbeat, we see the results we will derive for the variation in the tick rate of the light clock in S’ (as seen by the people in S) are not limited to the behavior of the light clock itself, but are actually statements about the way time itself is passing in S’ (as seen by the people in S).

Derivation of time dilation

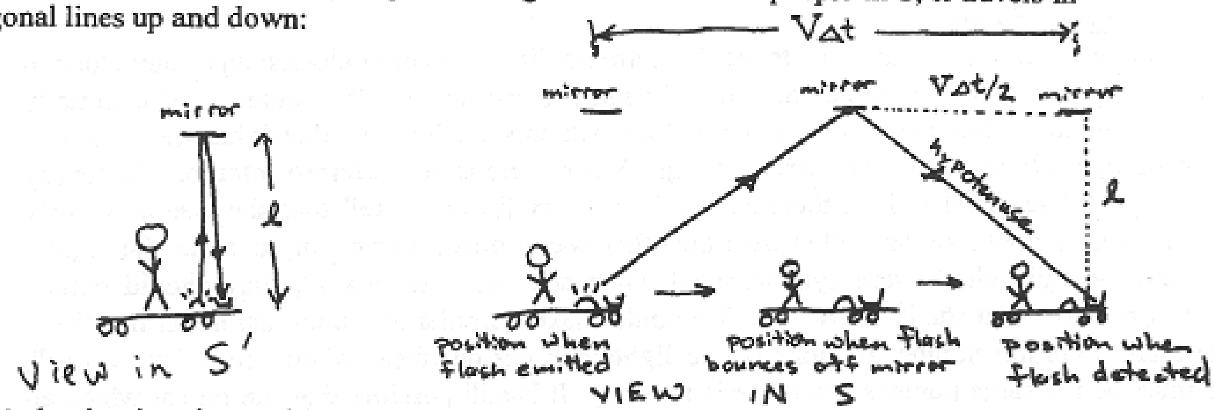
First, let's think about the path followed by the light, as seen in S' and then as seen in S. The situation is very similar to a person walking at constant speed who throws a ball straight up (as seen by the walking person) into the air, and then catches it. To the person who is walking, the ball goes straight up and then straight down. However, to a stationary observer, the ball follows a parabolic trajectory:



Now imagine that the moving person is moving rather quickly, and throws the ball straight up, but quite fast. In fact, the ball bounces off the ceiling and then back down. Again, as seen by the moving person the ball goes straight up, bounces, then comes straight back down. However, for the stationary person, the ball, if thrown very fast, travels almost in straight diagonal lines up and down:



Finally, let's look at the path of the light flash in the light clock. This is like a <u>very</u> fast ball. To the person in S', it goes straight up and straight down. For the people in S, it travels in diagonal lines up and down:



Let Δt be the time interval between when the flash is sent out and when it is received (i.e. the time interval between light clock ticks), as measured in S. Since the clock (and everything else in S') is moving at speed V, the light pulse must cover a horizontal distance of $V\Delta t$, as shown above. The total path length covered by the light is then the sum of the hypotenuses as shown above:

$$\text{path length (in S)} = 2\sqrt{\ell^2 + \left(\frac{V\Delta t}{2}\right)^2}$$

Since the light travels at speed c, the time that it takes to cover this path, which is equal to Δt, is given by

$$\Delta t = \frac{\text{path length (in S)}}{c} \Rightarrow (\Delta t)^2 = \frac{4}{c^2}\left[\ell^2 + \left(\frac{V\Delta t}{2}\right)^2\right] = \frac{4\ell^2}{c^2} + \frac{V^2}{c^2}(\Delta t)^2$$

However, it is also true that light travels with speed c in S', so we can use a similar method to find the time Δt' between ticks as measured in S':

$$\Delta t' = \frac{\text{path length (in S')}}{c} = \frac{2\ell}{c} \Rightarrow (\Delta t')^2 = \frac{4\ell^2}{c^2}$$

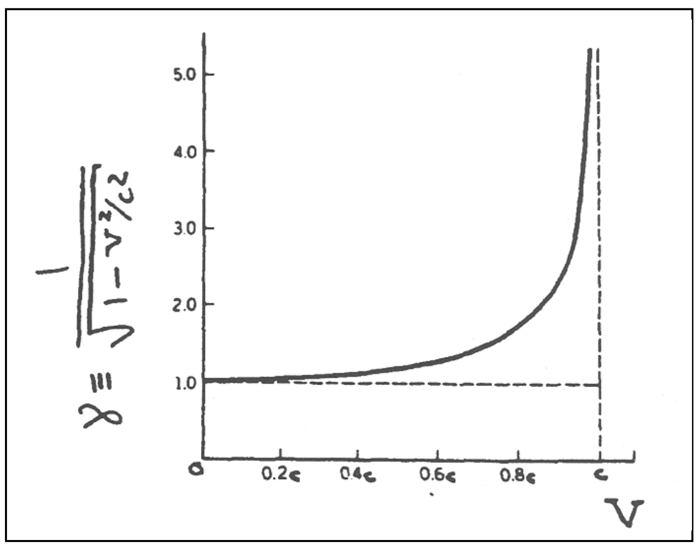
We can see right away that this is smaller than $(\Delta t)^2$, i.e. that the time between ticks as measured by the person in S' is shorter than the time between ticks as measured by the people in S! Let's get more quantitative. Substituting our expression for $(\Delta t')^2$ into the equation for $(\Delta t)^2$ gives

$$(\Delta t)^2 = (\Delta t')^2 + \frac{V^2}{c^2}(\Delta t)^2 \Leftrightarrow (\Delta t)^2\left(1 - \frac{V^2}{c^2}\right) = (\Delta t')^2 \Rightarrow \Delta t = \frac{1}{\sqrt{1 - V^2/c^2}}\Delta t'$$

We'll encounter that square-root factor a lot, so we define

$$\gamma \equiv \frac{1}{\sqrt{1 - V^2/c^2}} \Rightarrow \Delta t = \gamma \Delta t'$$

As we'll see eventually, V is always less than or equal to c, so γ is always greater than 1. This equation says that the time between ticks as measured in S is *greater than the time between ticks as measured in S'* by a factor of γ! The faster S' is moving, the greater the size of this effect. Let's assume that when everyone is at rest, their hearts beat at the same rate. Since he "dilation of time" derived above applies to all "clocks" in S', including the heartbeat of the person in S', this means that, as measured by the people in S. there is a longer time between the heartbeats of the person in S' than between their own heartbeats, and the faster she moves the longer this time becomes. Thus (according to the people in S), the person is S' is aging more slowly than they are!



This effect only become easily noticeable when V is greater than about 0.1c, as you can see from the plot of γ shown here.

Verification Of Time Dilation

[This paragraph is taken from a textbook.] A striking confirmation of time dilation was achieved in 1971 by an experiment carried out by J.C. Hafele and R.E. Keating. They transported very precise cesium-beam atomic clocks around the world on commercial jets. Since the speed of a jet plane is considerably less than c, the time-dilation effect is extremely small. However, the atomic clocks were accurate to about $\pm 10^{-9}$ s, so that the effect could be measured. The clocks were in the air for 45 hours, and their times were compared to reference atomic clocks kept on earth. The experimental results revealed that, within experimental error, the readings on the clocks on board the planes were different from those on earth by an amount that agreed with the prediction of relativity.

Proper time

We just showed that $\Delta t = \gamma\ \Delta t'$ However, this seems to contradict the fundamental postulate of relativity, since the equation is not symmetrical between the two reference frames; the time interval as measured in S is longer than that measured in S', and the faster S' goes, the more dramatic this effect becomes. However, there is something about the experiment with the light clock itself which makes a fundamental distinction between the two reference frames. The time interval Δt represents the time interval between two events: the first event is the flash, and the second is the reception of the flash. The fundamental distinction between the reference frames is that in S' these two events occur *at the same place*, while in S they occur at different places.

Thus, if we instead did the experiment with the light clock in S (instead of S') then the roles of the two reference frames would be reversed, and we would find $\Delta t' = \gamma\ \Delta t$, i.e. that the time as measured in S' between the two events is longer than the time as measured in S. So, the two reference frames really are equally good, it just depends on how we do the experiment.

We note that the shorter time is always measured in the reference frame in which the two events occur at the same place: When the clock was in S', the shorter time was measured in S', whereas when the clock was in S, the shorter time was measured in S. **We define the "proper time" between two events to be the time as measured in the reference frame in which the two events occur at the same place.** With this definition, we can summarize all such experiments with a single equation:

$$\boxed{\Delta t_{\text{other}} = \gamma \Delta t_{\text{p}}}$$

where Δt_{p} is the proper time, and Δt_{other} is the time as measured in some other reference frame. For example, if we do the experiment with the light clock in S, then both events (the flash and the reception) occur at the same place in S, so and $\Delta t_{\text{p}} = \Delta t$ and $\Delta t_{\text{other}} = \Delta t'$. However, if we do the experiment with the light clock in S' then both events (the flash and the reception) occur at the same place in S', so $\Delta \text{t}_{\text{p}} = \Delta t'$ and $\Delta t_{\text{other}} = \Delta t$.

The rate at which clocks tick

One important type of event is the ticking of any clock. Subsequent ticks of a clock occur at the same place in the clock's "rest frame" (the reference frame in which the clock isn't moving), so the people in this frame would measure Δt_{p} between the ticks. Observers in any other reference frame would thus measure a longer time between ticks, i.e., they would say that the clock runs slow. The conclusion of this is:

Clocks in the other person's reference frame run slow; the time between ticks is increased by the factor γ.

(This is equivalent to the equation form in the box above.)