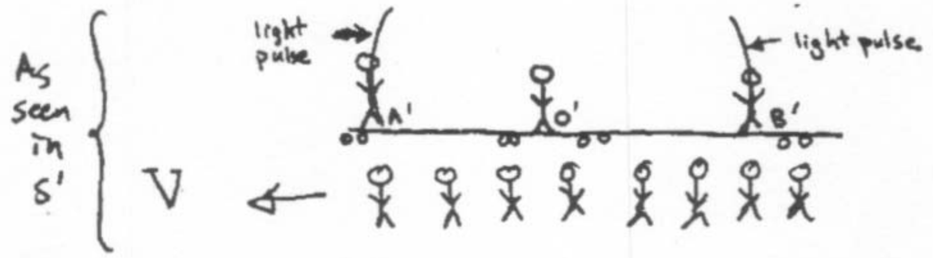
(Continuation of Relativity Notes by Walter F. Smith, Haverford College, 4-25-01)

SYNCHRONIZATION

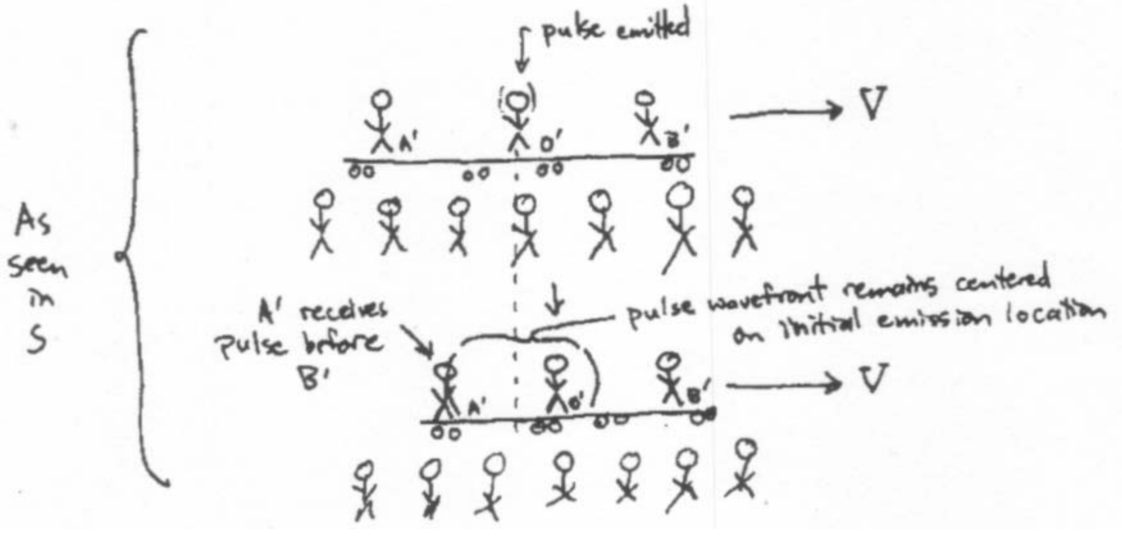
Our theory is not yet self-consistent. The S observers say that the clocks in S' run slow, and the S' observers say that the clocks in S run slow. How can both be right? The answer is that the two sets of observers don't agree on whether events are simultaneous or not!

To see this qualitatively, imagine that there are three equally spaced observers in S'. There are a great many observers spread out along the x-axis in S. To avoid any confusion that might occur because of delays associated with the propagation of light, each observer only reports on events that occur directly in front of him/her. After recording the times and positions of the events, the observers in each frame get together and compare notes.

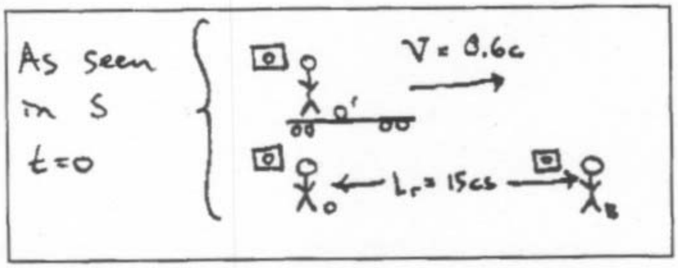
Observer O' emits a light flash. For this experiment, the two events of interest are the reception of the flash by observers A' and B'. In S', the flash spreads out toward A' at speed c and also toward B' at speed c, so these two events occur simultaneously, as shown below.



However, as seen in S, A' is moving toward the light flash, and so will reach it before B' who is moving away from the light flash. Thus the two events are not simultaneous:



This disagreement about simultaneity must be exactly big enough to resolve the time dilation dilemma mentioned above. Let's go back to a situation with just one person in S' and two in S. The situation as seen in S at t = 0 is as shown here. (All the observers have digital clocks, which are synchronized with the other clocks in their own frame.) The clock readings are as seen by the local observers, and we'll assume the clocks read in seconds. The distance between O and B is labelled L_r. This is the distance as measured in S, the frame in which O and B are both at rest, and so it's referred to as the "rest length". (So far as we know, the same length would be measured in S', but, as we'll see in a few pages, that's not really true.) L_r is measured in units of light seconds, abbreviated cs, which is the distance light would travel in one second: $1\,\text{cs} = c \cdot (1\,\text{s}) \cong 3_8\ \text{m}$.



The value shown for V makes γ come out to a nice round figure:

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{1}{\sqrt{\frac{25}{25}-\frac{9}{25}}} = \frac{1}{\sqrt{\frac{16}{25}}} = \frac{5}{4}$$

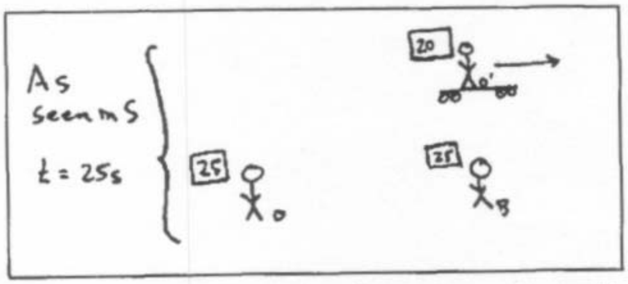
There are two important events: the first is when O' passes O (as shown above), and the second is when O' passes B. It's not hard to find the time Δt between these events (as measured in S), since this is just the time it takes for O' to move from O to B:

$$V = \frac{L_r}{\Delta t} \Leftrightarrow \Delta t = \frac{L_r}{V} = \frac{15\,\text{cs}}{\frac{3}{5}\text{c}} = 25\,\text{s}$$

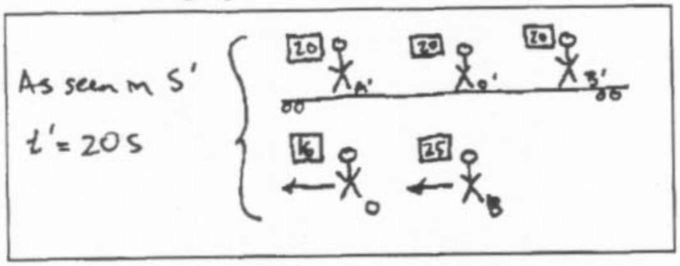
For observers in S', these two events occur at the same place (right in front of O'), so

$$\left.\begin{array}{r}\Delta t' = \Delta t_{\text{proper}} \text{ and } \Delta t = \Delta t_{\text{other}} \\ \Delta t_{\text{other}} = \gamma \Delta t_{\text{proper}}\end{array}\right\} \Rightarrow \Delta t = \gamma \Delta t' \Leftrightarrow \Delta t' = \frac{\Delta t}{\gamma} = \frac{25\,\text{s}}{\frac{5}{4}} = 20\,\text{s}$$

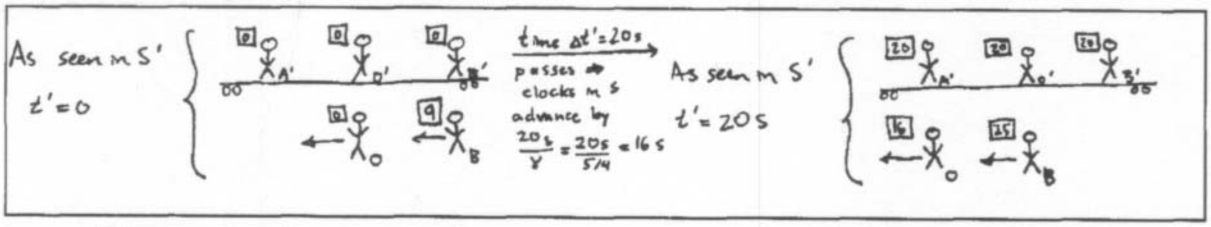
Therefore, when O' passes B, the clocks read as shown to the right.



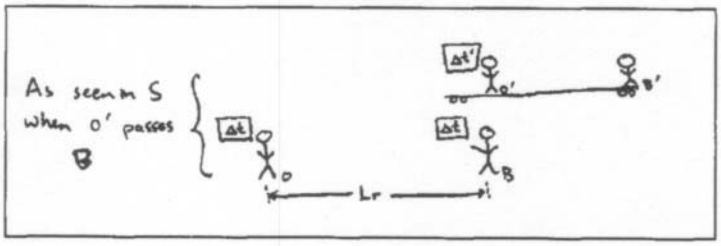
Now let's think about this from the point of view of O'. She says, "When I passed O, his clock read 0, and I know his clock runs slow by a factor $1/\gamma = 4/5$. So, 20 seconds later (when I pass B), I know that the clock belonging to O must read (20 s)(4/5) = 16 s. However, as I pass B, I can read his clock and it says 25 s. Restating this graphically, and adding a couple more observers in S', she says that the situation is as shown to the left. (Indeed, in accord with the reasoning of O', observer A' reports that the clock of



O reads 16 s.) Thus, according to the observers in S', the clocks of O and B are not correctly synchronized, but rather the clock of B leads that of O by (25 – 16) s = 9 s !!!!! This means that at $t' = 0$ (when O passes O'), the situation as seen in S' is as shown below.



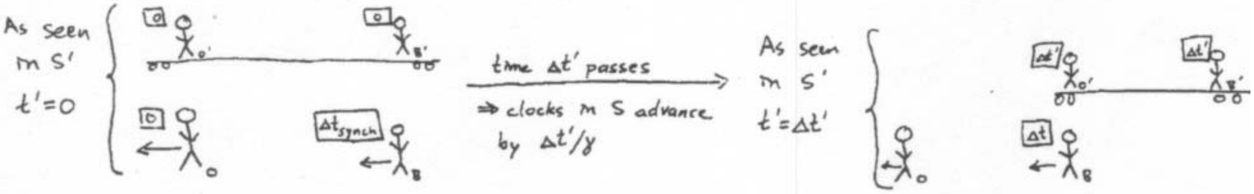
We can develop a general equation for the difference in synchronization between two clocks by repeating this whole scenario, using variables instead of numbers. The situation as seen in S is shown to the right.



As on the previous page, we have

$$V = \frac{L_r}{\Delta t} \Leftrightarrow \Delta t = \frac{L_r}{V} \quad \text{and} \quad \Delta t' = \frac{\Delta t}{\gamma}$$

The situation as seen in S' is shown here, with Δt_{synch} defined to be the difference in synchronization between the clocks of O and B:



As shown, at $t' = \Delta t'$, the observers in S' see a reading of Δt on the clock of B. They explain this reading with the following logic:

current reading = (initial reading) + (change in reading)

$$\Delta t = \qquad \Delta t_{synch} \qquad + \qquad \frac{\Delta t'}{\gamma}$$

We can combine this with the equations above to find Δt_{synch}:

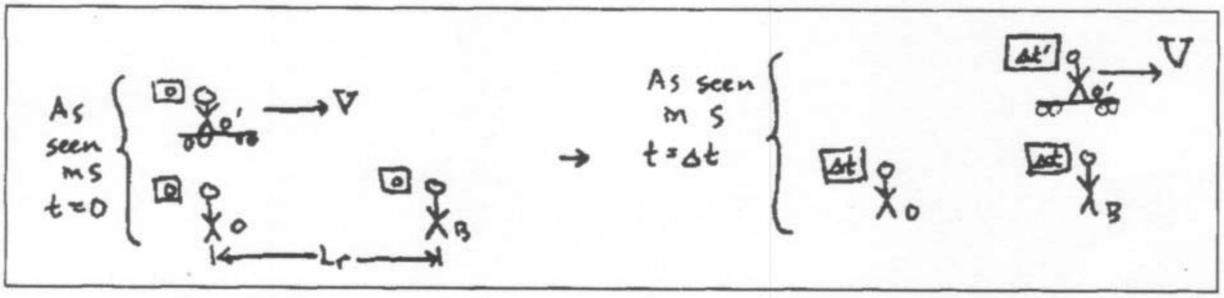
$$\left.\begin{aligned} \Delta t = \Delta t_{synch} + \frac{\Delta t'}{\gamma} \\ \Delta t' = \frac{\Delta t}{\gamma} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \gamma \equiv \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \\ \Delta t_{synch} = \Delta t\left(1-\frac{1}{\gamma^2}\right) \\ \Delta t = \frac{L_r}{V} \end{aligned}\right\} \Rightarrow \Delta t_{synch} = \frac{L_r}{V}\left[1-\left(1-\frac{V^2}{c^2}\right)\right] = \frac{L_r V}{c^2}$$

Which clock is ahead and which behind? Recall that, according to the people in S', when the clock at O reads 0, the clock at B reads >0, i.e. the clock at B leads the clock at O. To the people in S', the clock at B appears to be "chasing" after the clock at O, so,

$$\text{The chasing clock leads by } \Delta t_{synch} = \frac{L_r V}{c^2}$$

FITZGERALD CONTRACTION

There is still an inconsistency in our theory. Everything should be symmetrical between frames S and S'. Therefore, the people in S and S' should agree on the magnitude of their relative velocity (even though they may disagree about who's moving). Let's return to the situation discussed above:



The S observers can measure the velocity of S' by taking the distance travelled by O' and dividing it by the time interval, i.e. $V = \frac{L_r}{\Delta t}$. From the point of view of O', the people in S are moving to the left, so she can use a similar method to calculate their velocity, i.e. $V = \frac{L'}{\Delta t'}$, where L' is the distance between O and B as measured by observers in S'. (From what we know so far, we would expect that $L' = L_r$, but we'll soon see that this is wrong.) Again, the magnitude of V as measured by the people in S' must be the same as that measured in S, i.e.

$$\frac{L'}{\Delta t'} = \frac{L_r}{\Delta t}$$

As we showed before, for the situation shown above, $\Delta t = \gamma \Delta t'$. Substituting this in, we get

$$\frac{L'}{\Delta t'} = \frac{L_r}{\gamma \Delta t'} \Leftrightarrow L' = \frac{L_r}{\gamma}$$

In other words, the S' observers perceive the distance between O and B to be less than the S observers do!! As we did for the case of time dilation, we can make a more generally useful equation by using L_{other}:

$$L_{other} = \frac{L_r}{\gamma}$$

SUMMARY OF RELATIVISTIC EFFECTS

These are the three basic effects of special relativity. For convenience, we collect them here, along with a fourth useful equation

Time Dilation:	$\Delta t_{other} = \gamma \Delta t_p$
Synchronization:	The chasing clock leads by $\Delta t_{synch} = \frac{L_r V}{c^2}$
Fitzgerald Contraction:	$L_{other} = \frac{L_r}{\gamma}$
Velocity formula	v = distance/time (as long as all three are measured in the same frame)

THE LORENTZ TRANSFORMATION

The above equations are most useful when only one effect comes into play. However, there are many situations in which all three relativistic effects are important, and it can be confusing to apply the above equations correctly. To make things more straightforward, we will proceed to the relativistically correct version of the Galilean transformation, i.e. the set of equations which tell us the coordinates (in space and time) of an event in S if we know the coordinates in S'. It is clear that none of the three relativistic effects affect the y or z position of an object, so we have right away that $y = y'$ and $z = z'$, as was the case for the Galilean transformation. The equations for x and t are a little more difficult to obtain, so you will derive them on your next problem set. The result is the Lorentz transformation:

$$\begin{aligned} x &= \gamma (x' + Vt') \\ y &= y' \\ z &= z' \\ t &= \gamma (t' + x'V/c^2) \end{aligned}$$

These equations incorporate all the effects we've been discussing.

THE INVERSE LORENTZ TRANSFORM

It is easy to find the inverse transform: S and S' are completely symmetrical, except that, in S' the frame S is traveling with velocity $-V \Rightarrow$ all we do is interchange primed and unprimed variable, and substitute $-V$ for V:

$$\Rightarrow \boxed{\begin{aligned} x' &= \gamma\,(x - Vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - xV/c^2) \end{aligned}}$$

RECOVERY OF GALILEAN TRANSFORM FOR LOW SPEEDS

We know that the Galilean transform works quite well when things are moving at reasonable speeds, so we need to show that the Lorentz transform is consistent with this.

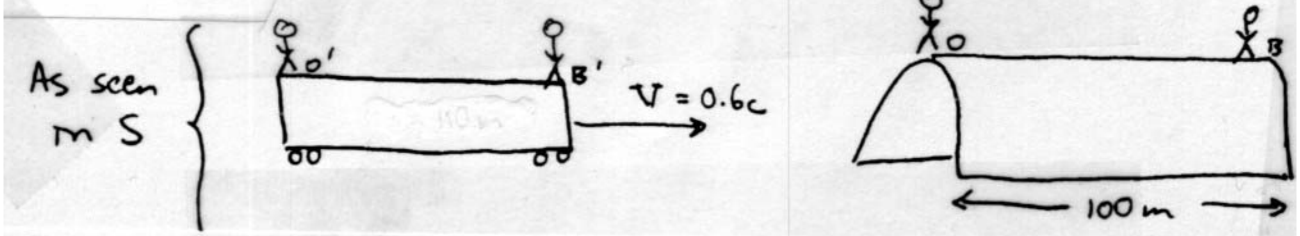
Recall that $\gamma = \dfrac{1}{\sqrt{1 - V^2/c^2}}$. For $V \ll$ c, this becomes $\gamma = \dfrac{1}{\sqrt{1-0}} = 1$. So, for $V \ll$ c, the Lorentz transform reduces to:

$$\begin{array}{lcl} x = \gamma(x' + Vt') & & x \cong x' + Vt' \\ y = y' & \xrightarrow{\gamma \approx 1} & y = y' \\ z = z' & & z = z' \\ t = \gamma(t' + x'\dfrac{V}{c^2}) & & t \cong t' \;+\; \underbrace{x'\dfrac{V}{c^2}}_{\approx 0 \text{ since } \frac{V}{c} \approx 0} \end{array}$$

So, we indeed recover the Galilean transform. In other words, the Galilean transform, as in all of Newtonian physics, is a special case of the Lorentz transform.

FAMOUS PARADOXES

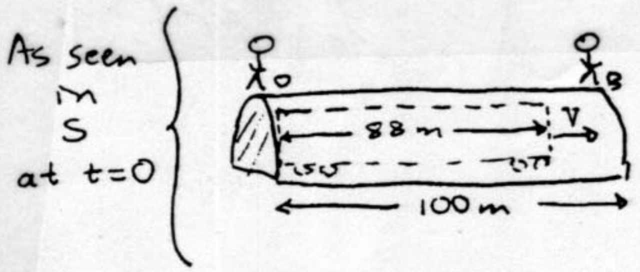
The train and tunnel paradox. A train of length 110 m (as measured by observers on the train) approaches a tunnel at speed V = 0.6c. The tunnel has a length of 100 m, as measured by stationary observers:



Let's call the reference frame of the tunnel S and that of the trains S'. The people in S' measure the proper length of the train ⇒ the length of the train in S is $L_{other} = L_p/\gamma$

$$\gamma \equiv \frac{1}{\sqrt{1-V^2/c^2}} = 1.25 \Rightarrow L_{other} = \frac{110m}{1.25} = 88m$$

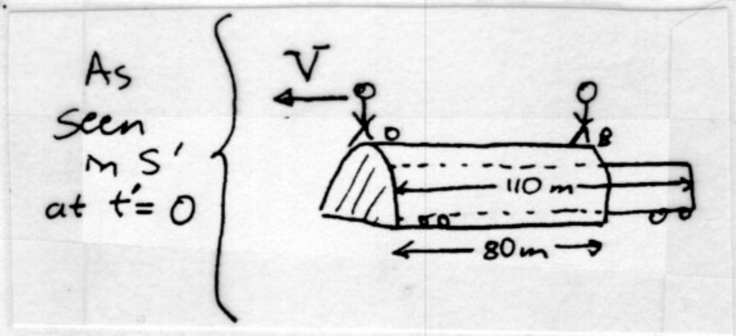
Thus, to people in S, the train is shorter than the tunnel. Observers O and B plan to briefly "catch" the train inside the tunnel. At the instant that O' passes O (call this t = t' = 0, as usual), both O and B will briefly slam down gates at each end of the tunnel, catching the train inside:



To avoid bloodshed, O and B will quickly re-open their doors after slamming them.

However, the people in S' see things very differently. The people in S measure the proper length of the tunnel ⇒the length of tunnel in S' is $L_{\text{other}} = L_p/\gamma = \frac{100\text{m}}{1.25} = 80$ m.

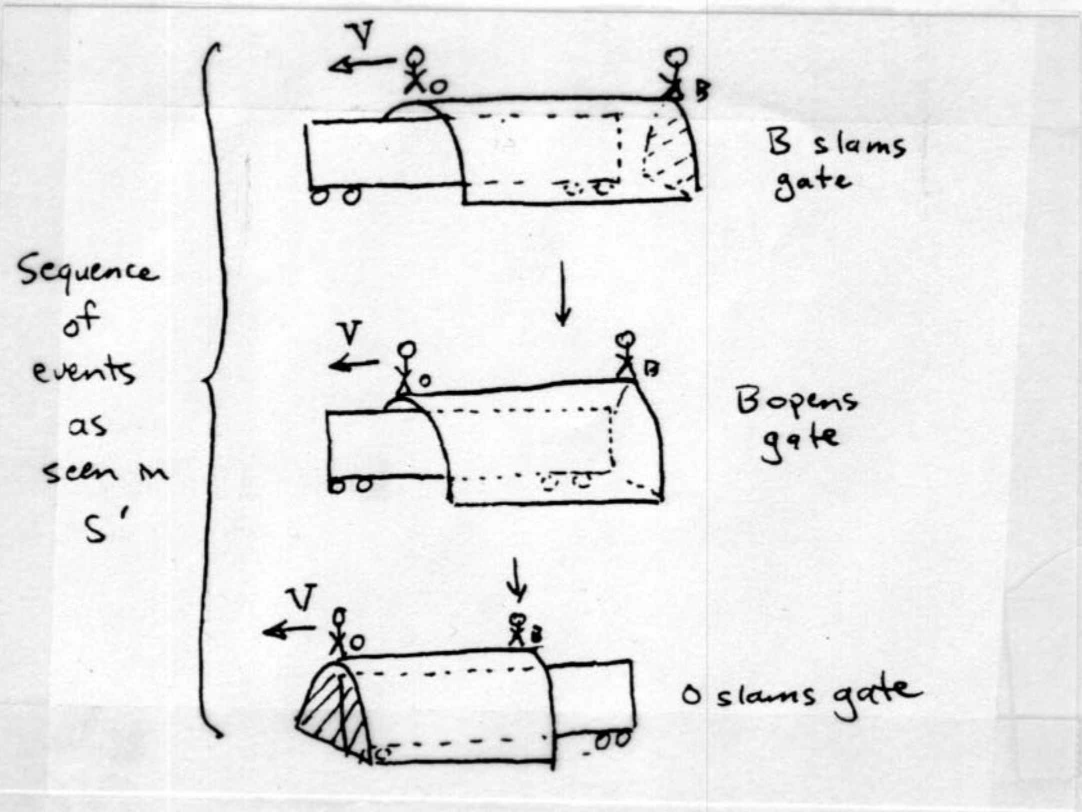
Since their train is 110 m long, there is no way that it can be "caught" in a tunnel that is only 80 m long. They agree that O slams her door at the instant O' passes, however, their view of the situation is:



So, who is right? Does the train get "caught" or not? The answer is that both are right, and whether or not the train is caught depends on your reference frame!

Qualitative Explanation

In S', O and B are moving to the left ⇒ B has the chasing clock ⇒ his clock leads that of O. Thus, at the instant B slams his gate (when his clock reads t = 0), the clock at O does not yet read t = 0, and O has not yet slammed her door. B then opens his door, the tunnel keeps moving to the left, and then O slams her gate:



Quantitative Treatment

Let x = 0 be the position of O and x' = 0 be the position (in S') of O'. Then O slams her gate at x = x' = 0 and t = t' = 0. What is the time and place (in S') that B slams his gate?

We'll use the Lorentz transform (in the inverse version):

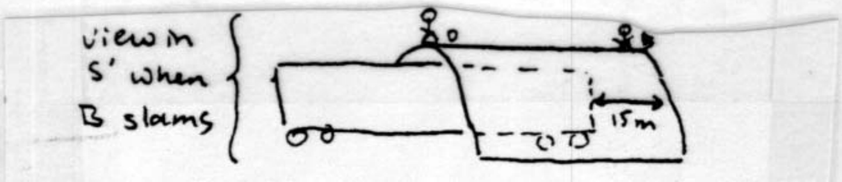
$$x' = \gamma\ (x - Vt)$$
$$t' = \gamma\ (t - xV/c^2)$$

The position of B is x = 100 in. (since the tunnel is 100 m long in S) and he slams his gate simultaneously with O, i.e., at t = 0

⇒ the position of B when he slams his gate is

$$x' = \gamma\,[100\text{ m} - V\,(0)] = 125\text{ m}$$

Since the position of B' is x' = 110 m (since the train is 110 m long), this means that B slams his gate 15 m in front of the train:



The time (in S') when B slams is

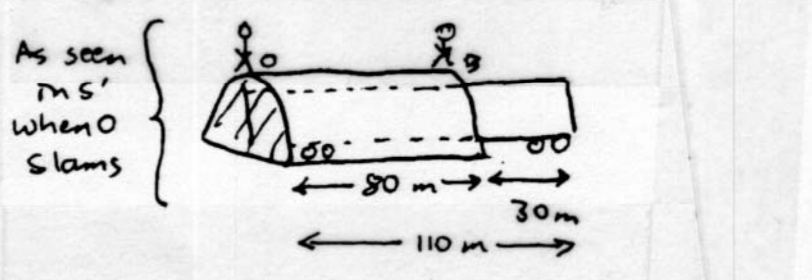
$$t' = \gamma\ (t - x\ V/c^2) = 1.25\ (0 - 100\ \text{m}\ (0.6\ c)/c^2)$$
$$= -2.5_{\ -7}\quad s = -0.25\ \mu\ s$$

Since O slams her gate at t' = 0, this means that B slams his gate 0.25 μ s before O does (as seen in S').

During this 0.25 μ s interval, the tunnel moves a distance of

$$0.25\ \ \mu\ s \times 0.6c = 45\ \text{m} \Rightarrow \text{the train now protrudes by } 45\ \text{m} - 15\ \text{m} = 30\ \text{m}$$

So, the situation when O slams is:



The Twin Paradox. Twin α stays on earth and sees twin β departing at V = 0.6c. Ten years later β returns, having spent half the trip traveling outbound at 0.6c, turning around upon reaching planet X, and then spending the second half of the trip returning to earth at 0.6c. As we know, α says that β will age more slowly, so that β will age less than α during the trip. However, β also says that α ages more slowly, so we might expect that β would say that α will age less during the trip. However, once β returns, we can compare them side by side, and we'd see that in fact β is the one who is younger. Doesn't this indicate that the reference frame of α is "preferred," in violation of the basic postulate of relativity?

Qualitative Explanation. Because β had to accelerate to turn around, he didn't stay in an inertial reference frame for the whole trip, while α did. Therefore the situation really isn't symmetrical, and this is why β ages less.

Quantitative Treatment.

To analyze this correctly, we need three reference frames:

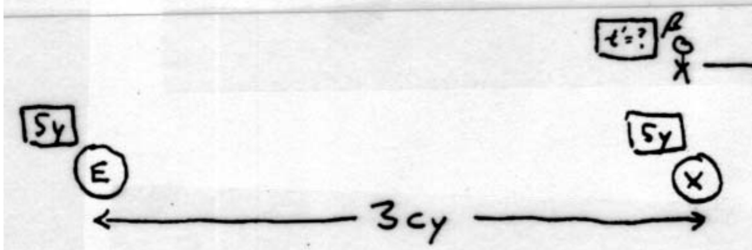
Earth ↔ S

$V = 0.6$ c ↔ S' (Twin β for first half of trip)

$V = -0.6$c ↔ S" (Twin β for second half)

In S: during the first 5y ("year"), B travels a distance (5y) (0.6c) = 3 cy ("lightyears")

So, as seen in S, the situation when β reaches planet X is:



(Displays of digital clocks shown in boxes)

What does the clock of β read when he reaches planet X?

We use the inverse Lorentz transform:

$$t' = \gamma(t - xV/c^2) = 1.25\left[5y - (3\cancel{c}y)(0.6\,\cancel{c})/\cancel{c^2}\right]$$

$$= 1.25\left[5y - 1.8y\right] = 4y$$

In other words, at this moment β says, "Only 4 y have passed. You people of planet X have mis-synchronized your clock with the one on earth. Although I can't see him, I know that his clock runs slow, so I think he's aged $\frac{4\text{ y}}{\gamma} = 3.2$y during my trip."

We can check the logic of β using the Lorentz transform to find the reading on the clock of α at $t' = 4$ y. Since we know x for α but not his position in S', it's easiest to use

$$t' = \gamma(t - xV/c^2) \qquad t = \frac{t'}{\gamma} + \frac{xV}{c^2} \qquad (1)$$

$$\longrightarrow x = 0 \text{ for } \alpha \Rightarrow t = \frac{t'}{\gamma} = \frac{4\ y}{1.25} = 3.2y$$ (in agreement with the logic of β).

Now (when he reaches planet X), β jumps from S' to S". When he arrives in S" the clock on planet X still reads 5 y (as it did just before his jump). However, the clocks in S" near planet X read

$$t'' = \gamma(t - xV/c^2) = 1.25\left[5y - (3\cancel{c}y)(-0.6\cancel{c})/\cancel{c^2}\right]$$

$$= 1.25\,[5y + 1.8y] = 8.5y$$

The new comrades of β in S" inform him that the clock of α now reads (using (1) above)

$$t = \frac{t''}{\gamma} + \underbrace{\frac{\cancel{x}V}{c^2}}_{\to 0 \text{ as above}} = \frac{8.5y}{\gamma} = 6.8y$$

This means that, if β accepts the reports of his comrades in S", he must accept that his twin has suddenly aged by 3.6y when he jumped from S' to S". NOTE: No one making local observations on α would see any sudden change in his age.

The return trip of β (in S") again takes 4 y as measured by β. During this trip, α again ages by $\frac{4\ y}{\gamma}$ = 3.2y as measured by observers in S".

⇒ Upon the return of β, he has aged by 4 y + 4 y = 8y.

However, α has aged by $\underbrace{3.2\ y}_{\text{trip out}} + \underbrace{3.6\ y}_{\text{Upon jump from S' to S''}} + \underbrace{3.2\ y}_{\text{trip back}} = 10\ y$ } this is the aging process perceived by β

So, when β returns both twins agree that β has aged 8 y while α has aged 10 y, but they disagree on the details of how this took place.

Reminder: This statement means that in S' the events of 1) β reaching planet X and 2) The clock of α reading are simultaneous. Of course, these events are not simultaneous in S. As seen in S, when β reaches planet X, the clock of α reads 5y, not 3.2y.